Exercise 17C: Solutions

1 Let $\backslash\text{bmit}a = \backslash\text{bmit}i - 4\backslash\text{bmit}j, \backslash\text{bmit}b = 2\backslash\text{bmit}i + 3\backslash\text{bmit}j$ and $\backslash\text{bmit}c = -2\backslash\text{bmit}i - 2\backslash\text{bmit}j$

a $\backslash\text{bmit}a \cdot \backslash\text{bmit}a = 17$

b $\backslash\text{bmit}b \cdot \backslash\text{bmit}b = 13$

c $\backslash\text{bmit}c \cdot \backslash\text{bmit}c = 8$

d $\backslash\text{bmit}a \cdot \backslash\text{bmit}b = -10$

e $\backslash\text{bmit}a \cdot (\backslash\text{bmit}b + \backslash\text{bmit}c) = (\backslash\text{bmit}i - 4\backslash\text{bmit}j) \cdot (\backslash\text{bmit}j) = -4$

f
$$\begin{aligned}&(\backslash\text{bmit}a + \backslash\text{bmit}b) \cdot (\backslash\text{bmit}a + \backslash\text{bmit}c)\\ &= \backslash\text{bmit}a \cdot \backslash\text{bmit}a + \backslash\text{bmit}a \cdot \backslash\text{bmit}c + \backslash\text{bmit}b \cdot \backslash\text{bmit}a + \backslash\text{bmit}b \cdot \backslash\text{bmit}c\\ &= 17 + 6 - 10 - 10\\ &= 3\end{aligned}$$

g
$$\begin{aligned}&\backslash\text{bmit}a + 2\backslash\text{bmit}b = 5\backslash\text{bmit}i + 2\backslash\text{bmit}j\\ &3\backslash\text{bmit}c - \backslash\text{bmit}b = -8\backslash\text{bmit}i - 9\backslash\text{bmit}j\\ &\therefore (\backslash\text{bmit}a + 2\backslash\text{bmit}b) \cdot (3\backslash\text{bmit}c - \backslash\text{bmit}b) = -58\end{aligned}$$

2 Let $\backslash\text{bmit}a = 2\backslash\text{bmit}i - \backslash\text{bmit}j, \backslash\text{bmit}b = 3\backslash\text{bmit}i - 2\backslash\text{bmit}j$ and $\backslash\text{bmit}c = -\backslash\text{bmit}i + 3\backslash\text{bmit}j$

a $\backslash\text{bmit}a \cdot \backslash\text{bmit}a = 5$

b $\backslash\text{bmit}b \cdot \backslash\text{bmit}b = 13$

c $\backslash\text{bmit}a \cdot \backslash\text{bmit}b = 8$

d $\backslash\text{bmit}a \cdot \backslash\text{bmit}c = -5$

e $\backslash\text{bmit}a \cdot (\backslash\text{bmit}a + \backslash\text{bmit}b) = 13$

3 $|\backslash\text{bmit}a| = 5$ and $|\backslash\text{bmit}b| = 6$

a
$$\begin{aligned}\backslash\text{bmit}a \cdot \backslash\text{bmit}b &= 5 \times 6\cos 45^\circ\\ &= 30 \times \frac{1}{\sqrt{2}}\\ &= 15\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}\backslash\text{bmit}a \cdot \backslash\text{bmit}b &= 5 \times 6\cos 135^\circ\\ &= 30 \times -\frac{1}{\sqrt{2}}\\ &= -15\sqrt{2}\end{aligned}$$

4 a
$$\begin{aligned}(\backslash\text{bmit}a + 2\backslash\text{bmit}b) \cdot (\backslash\text{bmit}a + 3\backslash\text{bmit}b) &= \backslash\text{bmit}a \cdot \backslash\text{bmit}a + 4\backslash\text{bmit}a \cdot \backslash\text{bmit}b + 4(\backslash\text{bmit}b \cdot \backslash\text{bmit}b)\\ &= |\backslash\text{bmit}a|^2 + 4\backslash\text{bmit}a \cdot \backslash\text{bmit}b + 4|\backslash\text{bmit}b|^2\end{aligned}$$

b
$$\begin{aligned}|\backslash\text{bmit}a + \backslash\text{bmit}b|^2 - |\backslash\text{bmit}a - \backslash\text{bmit}b|^2 &= (\backslash\text{bmit}a + \backslash\text{bmit}b) \cdot (\backslash\text{bmit}a + \backslash\text{bmit}b) + (\backslash\text{bmit}a - \backslash\text{bmit}b) \cdot (\backslash\text{bmit}a - \backslash\text{bmit}b)\\ &= (\backslash\text{bmit}a \cdot \backslash\text{bmit}a + 2\backslash\text{bmit}a \cdot \backslash\text{bmit}b + \backslash\text{bmit}b \cdot \backslash\text{bmit}b) - (\backslash\text{bmit}a \cdot \backslash\text{bmit}a - 2\backslash\text{bmit}a \cdot \backslash\text{bmit}b + \backslash\text{bmit}b \cdot \backslash\text{bmit}b)\\ &= 4\backslash\text{bmit}a \cdot \backslash\text{bmit}b\end{aligned}$$

c
$$\begin{aligned}\backslash\text{bmit}a \cdot (\backslash\text{bmit}a + \backslash\text{bmit}b) - \backslash\text{bmit}b(\backslash\text{bmit}a + \backslash\text{bmit}b &= \backslash\text{bmit}a \cdot \backslash\text{bmit}a + \backslash\text{bmit}a \cdot \backslash\text{bmit}b - \backslash\text{bmit}a \cdot \backslash\text{bmit}b + \backslash\text{bmit}b \cdot \backslash\text{bmit}a\\ &= |\backslash\text{bmit}a|^2 - |\backslash\text{bmit}b|^2\end{aligned}$$

d
$$\begin{aligned}\frac{\backslash\text{bmit}a \cdot (\backslash\text{bmit}a + \backslash\text{bmit}b) - \backslash\text{bmit}a \cdot \backslash\text{bmit}b}{|\backslash\text{bmit}a|} &= \frac{|\backslash\text{bmit}a|^2 + \backslash\text{bmit}a \cdot \backslash\text{bmit}b - \backslash\text{bmit}a \cdot \backslash\text{bmit}b}{|\backslash\text{bmit}a|}\\ &= |\backslash\text{bmit}a|\end{aligned}$$

5 a
$$\begin{aligned}\overrightarrow{AB} &= -2\backslash\text{bmit}i - 2\backslash\text{bmit}j - \backslash\text{bmit}i + 3\backslash\text{bmit}j\\ &= -3\backslash\text{bmit}i + \backslash\text{bmit}j\end{aligned}$$

b $|\overrightarrow{AB}| = \sqrt{9+1} = \sqrt{10}$

c
$$\begin{aligned}\backslash\text{bmit}a \cdot \overrightarrow{AB} &= |\backslash\text{bmit}a||\overrightarrow{AB}|\cos\theta\\ \therefore -4 &= \sqrt{10} \times 2\sqrt{2}\cos\theta\\ \therefore \cos\theta &= -\frac{4}{2\sqrt{20}}\end{aligned}$$

$$\therefore \theta = 116.57^\circ$$

6 $\overrightarrow{CD} = -\boldsymbol{c} + \boldsymbol{d}$

Let θ be the angle between $\boldsymbol{c}$ and $\boldsymbol{d}$

$$\boldsymbol{c} \cdot \boldsymbol{d} = |\boldsymbol{c}||\boldsymbol{d}| \cos\theta$$

$$\therefore \cos\theta = \frac{4}{5 \times 7}$$

Using the cosine rule.

$$\begin{aligned} |\overrightarrow{CD}|^2 &= 5^2 + 7^2 - 2 \times 5 \times 7 \cos\theta \\ &= 25 + 49 - 2 \times 5 \times 7 \times \frac{4}{35} \\ &= 66 \end{aligned}$$

$$\therefore |\overrightarrow{CD}| = \sqrt{66}$$

7 a $(\boldsymbol{i} + 2\boldsymbol{j}) \cdot (5\boldsymbol{i} + x\boldsymbol{j}) = -6$

$$\begin{aligned} 5 + 2x &= -6 \\ 2x &= -11 \\ x &= -\frac{11}{2} \end{aligned}$$

b $(x\boldsymbol{i} + 7\boldsymbol{j}) \cdot (-4\boldsymbol{i} + x\boldsymbol{j}) = 10$

$$\begin{aligned} -4x + 7x &= 10 \\ 3x &= 10 \\ x &= \frac{10}{3} \end{aligned}$$

c $(x\boldsymbol{i} + \boldsymbol{j}) \cdot (-2\boldsymbol{i} - 3\boldsymbol{j}) = x$

$$\begin{aligned} -2x - 3 &= x \\ -3 &= 3x \\ -1 &= x \end{aligned}$$

d $x(2\boldsymbol{i} + 3\boldsymbol{j}) \cdot (\boldsymbol{i} + x\boldsymbol{j}) = 6$

$$\begin{aligned} x(2 + 3x) &= 6 \\ 2x + 3x^2 &= 6 \\ 3x^2 + 2x - 6 &= 0 \\ x &= \frac{-2 \pm \sqrt{76}}{6} \end{aligned}$$

8 a

$$\begin{aligned} \overrightarrow{AP} &= \overrightarrow{AO} + \overrightarrow{OP} \\ &= -4\boldsymbol{i} - 4\boldsymbol{j} + q(2\boldsymbol{i} + 5\boldsymbol{j}) \\ &= (2q - 1)\boldsymbol{i} + (5q - 4)\boldsymbol{j} \\ &= 2q\boldsymbol{i} + 5q\boldsymbol{j} - (\boldsymbol{i} + 4\boldsymbol{j}) \\ &= -\boldsymbol{a} + q\boldsymbol{b} \end{aligned}$$

b

$$\begin{aligned} \overrightarrow{AP} \cdot \overrightarrow{OB} &= 0 \\ \Rightarrow ((2q - 1)\boldsymbol{i} + (5q - 4)\boldsymbol{j}) \cdot (2\boldsymbol{i} + 5\boldsymbol{j}) &= 0 \\ \Rightarrow 4q - 2 + 25q - 20 &= 0 \\ \Rightarrow 29q - 22 &= 0 \\ \Rightarrow q &= \frac{22}{29} \end{aligned}$$

c $\overrightarrow{OP} = q\boldsymbol{b} = \frac{22}{9}(2\boldsymbol{i} + 5\boldsymbol{j})$

Cooordinates of P are $\left(\frac{44}{29}, \frac{110}{29}\right)$

9 a

$$\begin{aligned} (\boldsymbol{i} + 2\boldsymbol{j}) \cdot (\boldsymbol{i} - 4\boldsymbol{j}) &= \sqrt{5} \times \sqrt{17} \cos\theta \\ -7 &= \sqrt{85} \cos\theta \\ \cos\theta &= -\frac{7}{\sqrt{85}} \\ \theta &= 139.40^\circ \end{aligned}$$

b

$$\begin{aligned} -2\boldsymbol{i} + \boldsymbol{j}) \cdot (-2\boldsymbol{i} - 2\boldsymbol{j}) &= \sqrt{5} \times \sqrt{8} \cos\theta \\ 2 &= \sqrt{40} \cos\theta \\ \cos\theta &= \frac{2}{\sqrt{40}} \end{aligned}$$

$$\theta = 71.57^\circ$$

c $2\backslash\mathrm{bmit}i - \backslash\mathrm{bmit}j) \cdot (4\backslash\mathrm{bmit}i = \sqrt{5} \times 4\cos\theta$

$$8 = 4\sqrt{5}\cos\theta$$

$$\cos\theta = \frac{2}{\sqrt{5}}$$

$$\theta = 26.57^\circ$$

d $7\backslash\mathrm{bmit}i + \backslash\mathrm{bmit}j) \cdot (-\backslash\mathrm{bmit}i + \backslash\mathrm{bmit}i) + = \sqrt{50} \times \sqrt{2}\cos\theta$

$$-6 = 10\cos\theta$$

$$\cos\theta = -\frac{3}{5}$$

$$\theta = 126.87^\circ$$

10 $\backslash\mathrm{bmit}a \cdot \backslash\mathrm{bmit}b = |\backslash\mathrm{bmit}a||\backslash\mathrm{bmit}b|\cos\theta$

If $\backslash\mathrm{bmit}a$ and $\backslash\mathrm{bmit}a$ are non-zero vectors, then

$\backslash\mathrm{bmit}a \cdot \backslash\mathrm{bmit}b = 0 \Leftrightarrow \cos\theta = 0$

11a $\vec{OM} = \vec{OA} + \vec{AM}$

$$= \backslash\mathrm{bmit}a + \frac{1}{2}(\backslash\mathrm{bmit}b - \backslash\mathrm{bmit}a)$$

$$= \frac{1}{2}(\backslash\mathrm{bmit}a + \backslash\mathrm{bmit}b)$$

$$= \frac{3}{2}\backslash\mathrm{bmit}i$$

b $\backslash\mathrm{bmit}a \cdot \vec{OM} = |\backslash\mathrm{bmit}a|\vec{OM}|\cos(\angle AOM)$

$$\cos(\angle AOM) = \frac{\frac{3}{2}}{\sqrt{2} \times \frac{3}{2}}$$

$$\therefore \angle AOM = 45^\circ$$

c $\vec{MB} \cdot \vec{MO} = |\vec{MB}||\vec{MO}|\cos(\angle BMO)$

$$\cos(\angle BMO) = \frac{-\frac{3}{4}}{\frac{\sqrt{5}}{2} \times \frac{3}{2}}$$

$$\cos(\angle BMO) = -\frac{1}{\sqrt{5}}$$

$$\angle BMO = 116.57^\circ$$

12a i $\vec{OM} = \vec{OA} + \vec{AM}$

$$= \backslash\mathrm{bmit}a + \frac{1}{2}(\backslash\mathrm{bmit}b - \backslash\mathrm{bmit}a)$$

$$= \frac{1}{2}(\backslash\mathrm{bmit}a + \backslash\mathrm{bmit}b)$$

$$= \frac{1}{2}(3\backslash\mathrm{bmit}i + 4\backslash\mathrm{bmit}j)$$

ii $\vec{ON} = \vec{OA} + \vec{AN}$

$$= \backslash\mathrm{bmit}a + \frac{1}{2}(\backslash\mathrm{bmit}c - \backslash\mathrm{bmit}a)$$

$$= \frac{1}{2}(\backslash\mathrm{bmit}c + \backslash\mathrm{bmit}b)$$

$$= \frac{1}{2}(\backslash\mathrm{bmit}i + 6\backslash\mathrm{bmit}j)$$

b $\vec{OM} \cdot \vec{ON} = |\vec{ON}||\vec{OM}|\cos(\angle MON)$

$$\cos(\angle MON) = \frac{\frac{27}{4}}{\frac{5}{2} \times \frac{\sqrt{37}}{2}}$$

$$\cos(\angle MON) = \frac{1}{\sqrt{5}}$$

$$\angle BMO = 27.41^\circ$$

c $\vec{OM} \cdot \vec{OC} = |\vec{OM}||\vec{OC}|\cos(\angle MOC)$

$$\cos(\angle MOC) = \frac{9}{\frac{5}{2} \times \sqrt{40}}$$

$$\cos(\angle MOC) = \frac{9}{5\sqrt{10}}$$

$$\angle BMO = 55.30^\circ$$